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A supersymmetric dispersive water wave equation

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Abstract. A supersymmetric extension of the classical dispersive water wave equation is proposed. The system is shown to be tri-Hamiltonian. In particular, the second Hamiltonian structure is analogous to the $N = 2$ superconformal algebra. A gauge map to the even-parity super KP system is exhibited.

1. Introduction

The classical dispersive water wave equation has been known for a long time. By looking upon this equation as the first in a hierarchy of equations, Kupershmidt found the system to be integrable. There were other interesting results. The hierarchy has a tri-Hamiltonian structure and is the first among a new kind of integrable systems which have come to be known in the literature as non-standard integrable systems [1]. Today, the fields involved in the classical dispersive water wave equation form the basis of the two-boson realization of $W_{1+\infty}$ and $\hat{W}_\infty$ algebras. In this paper, we start with a superfield version of the pseudo-differential operator associated with the aforesaid hierarchy. At this point it may be noted that the usual supersymmetrization procedure of replacing all the derivatives '∂' by their counterpart 'D' does not lead always to equations which will give back the ordinary system as a special case. An important example is the KP hierarchy [3]. On the other hand, the even-order SKP (super KP) equation does possess all these nice features. Keeping this in mind we have considered a Lax operator of the form $L = D^2 + u + hD^{-1}$. The resulting second flow reduces to the classical one when the odd fields are set to zero. We next study the Hamiltonian structures proceeding à la Kupershmidt. The second Hamiltonian structure is analogous to the classical $N = 2$ superconformal algebra [4]. The third Hamiltonian structure turns out to be non-local. It has been known for sometime that the equations of motion in quantum 2D gravity can be formulated in terms of integrable nonlinear equations of the KdV type. Since all the KdV hierarchies are contained in the larger integrable system, i.e. the KP hierarchy, it has been conjectured that it provides a universal framework exhibiting the underlying structure of 2D quantum gravity. The even-parity superlax operator appears to play a similar role in quantum supergravity and our system is a special case—a constrained one—of the even-order super KP system considered by Watanabe [5]. Finally, by a gauge transformation, we arrive at the even-parity pseudo-differential operator of the super KP hierarchy [6].

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2. The dispersive water wave hierarchy

The classical dispersive water wave equation studied by Kupershmidt is of the form

$$u_t = \tfrac{1}{2}(u^2 + 2h - u_x)_x$$
$$h_t = \tfrac{1}{2}(2uh + h_x)_x. \tag{1}$$

He found this to be the first (with $P = \frac{1}{2}L^2$) member of the hierarchy given by

$$L_t = [(P^+)_{\geqslant 1})^+, L] = [L, ((P^+)_{\leqslant 0})^+] \tag{2}$$

with

$$L = \partial + u + h\partial^{-1} \qquad P = \sum_s \partial^s p_s(m) \tag{3}$$

and $P \in z(L)$, the centralizer of L in the ring of pseudo-differential operators. P^+ is the adjoint of P given by $P^+ = \sum_s (-1)^s p_s(m)\partial^s$.

Hierarchy (2) is a tri-Hamiltonian system. For $P = L^m$, it can be written in the form

$$\begin{pmatrix} u_t \\ h_t \end{pmatrix} = B^{\mathrm{I}}\delta H_{m+1} = B^{\mathrm{II}}\delta H_m = B^{\mathrm{III}}\delta H_{m-1} \tag{4}$$

where

$$\delta H = \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta h} \end{pmatrix}$$

$$B^{\mathrm{I}} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \tag{5}$$

$$B^{\mathrm{II}} = \begin{pmatrix} 2\partial & \partial u - \partial^2 \\ u\partial + \partial^2 & h\partial + \partial h \end{pmatrix} \tag{6}$$

$$B^{\mathrm{III}} = \begin{pmatrix} 2(u\partial + \partial u) & 2(h\partial + \partial h) + \partial(u - \partial)^2 \\ 2(h\partial + \partial h) + (u + \partial)^2\partial & (u + \partial)(h\partial + \partial h) + (h\partial + \partial h)(u - \partial) \end{pmatrix}. \tag{7}$$

3. The superfield extension

We shall consider the hierarchy

$$L_t = [L, ((P^+)_{\leqslant 0})^+] = [((P^+)_{\geqslant 1})^+, L] \tag{8}$$

with $L = D^2 + u + hD^{-1}$, u being an even superfield and h odd. $(D = \partial_\theta + \theta\partial_x)$. With $P = \frac{1}{2}L^2$, we get the following equations:

$$u_t = \tfrac{1}{2}(-u^{(4)} + 2uu^{(2)} + 2h^{(3)}) \tag{9}$$
$$h_t = \tfrac{1}{2}(h^{(4)} + 2uh^{(2)} + 2u^{(2)}h). \tag{10}$$

Let $u = u_1 + \theta\phi_1$ where u_1 is even and ϕ_1 is odd and $h = \psi_1 + \theta h_1$ where h_1 is even and ψ_1 is odd ($D^n(f) = f^{(n)}$ denotes the nth-super derivative of f). Then equations (9) and (10) are, in component form,

$$u_{1,t} = \tfrac{1}{2}(-u_{1,x} + u_1^2 + 2h_1)_x \tag{11}$$
$$\phi_{1,t} = \tfrac{1}{2}(-\phi_{1,x} + 2u_1\phi_1 + 2\psi_{1,x})_x \tag{12}$$
$$\psi_{1,t} = \tfrac{1}{2}(\psi_{1,x} + 2u_1\psi_1)_x \tag{13}$$
$$h_{1,t} = \tfrac{1}{2}(h_{1,x} + 2u_1h_1 + 2\phi_1\psi_1)_x. \tag{14}$$

Note that setting $\phi_1 = 0$, $\psi_1 = 0$ gives equations for u_1 and h_1 identical to equation (1).

Let us now write down a general form of (8). With $\mathrm{P}=\mathrm{L}^m=\sum_s \mathrm{D}^s p_s(m)$

$$[L, ((P^+)_{\leqslant 0})^+]_{\geqslant -1} = p_0^{(2)}(m) - p_{-1}^{(2)}(m)\mathrm{D}^{-1} + \cdots$$

$$u_t = p_0^{(2)}(m) \qquad h_t = -p_{-1}^{(2)}(m) \tag{15}$$

$$p_0(m) = \frac{\delta H_{m+1}}{\delta h} \qquad p_{-1}(m) = \frac{\delta H_{m+1}}{\delta u}. \tag{16}$$

4. The Hamiltonian structures

If $L^m = \sum_s \mathrm{D}^s p_s(m)$, the corresponding adjoint operator is defined to be [2]

$$(L^+)^m = \sum_s (-1)^{[s/2]} p_s(m)\mathrm{D}^s$$

where $[S/2]$ is the largest integer which does not exceed $S/2$. In defining the adjoint operator, we have simply supersymmetrized the definition of Kupershmidt in [1]. The same definition has also been adopted in [2].

Let us use the identity

$$(L^+)^{m+1} = L^+(L^+)^m \tag{16a}$$

which, written in full, leads to the following

$$\sum_s (-1)^{[s/2]} p_s(m+1)\mathrm{D}^s = (-\mathrm{D}^2 + u - \mathrm{D}^{-1}h) \sum_s (-1)^{[s/2]} p_s(m)\mathrm{D}^s.$$

Equating the coefficients of D^0, D^{-1}, D^{-2} and D^{-3} on both sides, we obtain the following recurrence relations:

$$\begin{aligned} p_0(m+1) = {} & -p_0^{(2)}(m) + p_{-2}(m) + up_0(m) \\ & + \sum_{j=0}^{\infty} \{hp_{2j+2}(m)\}^{(2j+1)} - \sum_{j=0}^{\infty} \{hp_{2j+1}(m)\}^{(2j)} \end{aligned} \tag{17}$$

$$\begin{aligned} -p_{-1}(m+1) = {} & p_{-1}^{(2)}(m) - p_{-3}(m) - up_{-1}(m) \\ & - \sum_{j=0}^{\infty} \{hp_{2j+1}(m)\}^{(2j+1)} + \sum_{j=0}^{\infty} \{hp_{2j}(m)\}^{(2j)} \end{aligned} \tag{18}$$

$$\begin{aligned} -p_{-2}(m+1) = {} & p_{-2}^{(2)}(m) - p_{-4}(m) - up_{-2}(m) \\ & - \sum_{j=0}^{\infty} \{hp_{2j}(m)\}^{(2j+1)} + \sum_{j=0}^{\infty} \{hp_{2j-1}(m)\}^{(2j)} \end{aligned} \tag{19}$$

$$\begin{aligned} p_{-3}(m+1) = {} & -p_{-3}^{(2)}(m) + p_{-5}(m) + up_{-3}(m) + \\ & \sum_{j=0}^{\infty} \{hp_{2j-1}(m)\}^{(2j+1)} = \sum_{j=0}^{\infty} \{hp_{2j-2}(m)\}^{(2j)}. \end{aligned} \tag{20}$$

On the other hand we can also use the following relation which is nothing other than (16a) written in reverse order,

$$(L^+)^{m+1} = (L^+)^m L^+$$

or

$$\sum_s (-1)^{[s/2]} p_s(m+1)\mathrm{D}^s = \sum_s (-1)^{[s/2]} p_s(m)\mathrm{D}^s(-\mathrm{D}^2 + u - \mathrm{D}^{-1}h).$$

Equating the coefficients of D^{-1}, D^{-2} and D^{-3} on both sides we obtain another set of recurrence relations:

$$-p_{-1}(m+1) = -p_{-3}(m) - p_{-1}(m)u + p_0(m)h \tag{21}$$

$$-p_{-2}(m+1) = -p_{-4}(m) - p_{-1}(m)u^{(1)} - p_{-2}(m)u - p_0(m)h^{(1)} + p_{-1}(m)h \tag{22}$$

$$p_{-3}(m+1) = p_{-5}(m) + p_{-1}(m)u^{(2)} + p_{-3}(m)u - p_0(m)h^{(2)} - p_{-2}(m)h. \tag{23}$$

Eliminating $p_{-2}(m)$, $p_{-3}(m)$, $p_{-4}(m)$ and $p_{-5}(m)$ from these equations, we finally get

$$p_0^{(2)}(m+1) = (-\mathrm{D}^4 + u\mathrm{D}^2 - h\mathrm{D} + u^{(2)})p_0(m) + (u^{(1)} - 2h - 2\mathrm{D}^3)p_{-1}(m) \tag{24}$$

$$-p_{-1}^{(2)}(m+1) = (2h\mathrm{D}^2 + h^{(2)})p_0(m) + (-\mathrm{D}^4 - u\mathrm{D}^2 + h\mathrm{D} - h^{(1)})p_{-1}(m). \tag{25}$$

So we may now write the Hamiltonian structures as

$$\begin{pmatrix} u_t \\ h_t \end{pmatrix} = B^{\mathrm{I}} \begin{pmatrix} p_{-1}(m) \\ p_0(m) \end{pmatrix} = B^{\mathrm{II}} \begin{pmatrix} p_{-1}(m-1) \\ p_0(m-1) \end{pmatrix} \tag{26}$$

with

$$B^{\mathrm{I}} = \begin{pmatrix} 0 & \mathrm{D}^2 \\ -\mathrm{D}^2 & 0 \end{pmatrix} \tag{27}$$

$$B^{\mathrm{II}} = \begin{pmatrix} u^{(1)} - 2h - 2\mathrm{D}^3 & -\mathrm{D}^4 + u\mathrm{D}^2 - h\mathrm{D} + u^{(2)} \\ -\mathrm{D}^4 - u\mathrm{D}^2 - h^{(1)} + h\mathrm{D} & 2h\mathrm{D}^2 + h^{(2)} \end{pmatrix}. \tag{28}$$

So equations (27) and (28) yield the first and second Hamiltonian structure of the equations (11)–(14). One may note that since $\mathrm{D}^2 = \partial$, the first symplectic form is identical to that of the ordinary dispersive water wave equation. The second symplectic operator, in the limit when all the odd variables go to zero, reproduces that of Kupershmidt. We can now proceed to determine the third symplectic operator in the present case. We first of all note that the evolution of the superfields (u, h) can also be written as follows

$$u_t = (u^{(1)} - 2h - 2\mathrm{D}^3)p_{-1}(m-1) + (-\mathrm{D}^4 + \mathrm{D}^2u - h\mathrm{D})p_0(m-1) \tag{24a}$$

$$h_t = (-\mathrm{D}^4 - u\mathrm{D}^2 - \mathrm{D}h)p_{-1}(m-1) + (\mathrm{D}^2h + h\mathrm{D}^2)p_0(m-1) \tag{25a}$$

and using relations (24) and (25) and their derivatives (with m replaced by $m-2$), we can finally write

$$\begin{pmatrix} u_t \\ h_t \end{pmatrix} = B^{\mathrm{I}} \begin{pmatrix} p_{-1}(m) \\ p_0(m) \end{pmatrix} = B^{\mathrm{II}} \begin{pmatrix} p_{-1}(m-1) \\ p_0(m-1) \end{pmatrix} = B^{\mathrm{III}} \begin{pmatrix} p_{-1}(m-2) \\ p_0(m-2) \end{pmatrix}$$

with

$$B^{\mathrm{III}} = \begin{pmatrix} \begin{matrix} -4u\mathrm{D}^3 - 2u^{(1)}\mathrm{D}^2 - 2u^{(2)}\mathrm{D} \\ +u(u^{(1)} - 2h) - u^{(3)} + u^{(1)}\mathrm{D}^{-1}h \\ -h\mathrm{D}^{-1}u^{(1)} + (u^{(1)} - 2h)\mathrm{D}^{-2}u\mathrm{D}^2 \\ +u^{(2)}\mathrm{D}^{-2}(u^{(1)} - 2h) \end{matrix} & \begin{matrix} \mathrm{D}^6 - 2u\mathrm{D}^4 - 2h\mathrm{D}^3 \\ +(4h^{(1)} - 3u^{(2)} + u^2)\mathrm{D}^2 \\ -(h^{(2)} + 2uh)\mathrm{D} \\ -u^{(4)} + 2u^{(2)}u + 2h^{(3)} \\ +h\mathrm{D}^{-1}h\mathrm{D} - (u^{(1)} - 2h)\mathrm{D}^{-2}h\mathrm{D}^2 \end{matrix} \\ \begin{matrix} -\mathrm{D}^6 - 2u\mathrm{D}^4 - 2h\mathrm{D}^3 \\ -(u^2 + 2h^{(1)} + u^{(2)})\mathrm{D}^2 \\ +(-h^{(2)} + uh)\mathrm{D} - h^{(3)} - uh^{(1)} + 2hu^{(1)} \\ -h^{(1)}\mathrm{D}^{-2}u\mathrm{D}^2 + h^{(2)}\mathrm{D}^{-2}(u^{(1)} - 2h) \\ -h^{(1)}\mathrm{D}^{-1}h + h\mathrm{D}^{-1}u\mathrm{D}^2 \end{matrix} & \begin{matrix} -u^{(2)}\mathrm{D}^{-2}h\mathrm{D}(2h^{(2)} + 4uh)\mathrm{D}^2 \\ +h^{(4)} + 2uh^{(2)} + 2hu^{(2)} \\ +h^{(1)}\mathrm{D}^{-2}h\mathrm{D}^2 \\ -h\mathrm{D}^{-1}h\mathrm{D}^2 - h^{(2)}\mathrm{D}^{-2}h\mathrm{D} \end{matrix} \end{pmatrix}. \tag{29}$$

5. $N = 2$ superconformal algebra

Let us now consider the second Hamiltonian structure B^{II} more carefully. Writing out the superfields in terms of the components

$$u = u_1 + \theta\phi_1 \qquad h = \psi_1 + \theta h_1$$

we can evaluate the Poisson or super Poisson brackets from equations (24*a*), (25*a*). For example, consider

$$u_t = B^{\mathrm{II}}_{11}\frac{\delta H_m}{\delta u} + B^{\mathrm{II}}_{12}\frac{\delta H_m}{\delta h}$$
$$h_t = B^{\mathrm{II}}_{21}\frac{\delta H_m}{\delta u} + B^{\mathrm{II}}_{22}\frac{\delta H_m}{\delta h}$$

where B^{II}_{ij} are the elements of the matrix B^{II}. Now from the definition of the symplectic operator B^{II} we get

$$\begin{aligned}\{u(x), u(y)\}_2 &= B^{\mathrm{II}}_{11}\Delta(x-y)\\ &= (u^{(1)} - 2h - 2\mathrm{D}^3)\Delta(x-y)\end{aligned}$$

where $\Delta(x-y)$ is the super delta function

$$\Delta(x-y) = (\theta_1 - \theta_2)\delta(x-y).$$

Also,

$$\begin{aligned}\{u(x), h(y)\}_2 &= B^{\mathrm{II}}_{12}\Delta(x-y)\\ &= (-\mathrm{D}^4 + u\mathrm{D}^2 - h\mathrm{D} + u^{(2)})\Delta(x-y).\end{aligned}$$

Recalling our definition on page 2854 of u and h

$$u = u_1 + \theta\phi_1 \qquad h = \psi_1 + \theta h_1$$

we get

$$u^{(1)} = 2h = \theta_1 u_{1,x} + \psi_1 = 2\psi_2 - 2\theta_1 u_2.$$

So,

$$\begin{aligned}(u^{(1)} - 2h - 2\mathrm{D}^3)\Delta(x-y) &= -2\partial_x\delta(x-y) - \theta_1(\psi_1 - 2\psi_2)\delta(x-y)\\ &\quad +\theta_2(\psi_1 - 2\psi_2)\delta(x-y) + \theta_1\theta_2(-u_{1,x} + 2u_2 + 2\partial_x^2)\delta(x-y)\\ &= \{u_1(x), u_1(y)\}_2 + \theta_2\{u_1(x)_1\psi_1(y)\}_2 + \theta_1\{\psi_1(x), u_1(y)\}_2\\ &\quad -\theta_1\theta_2\{\psi_1(x), \psi_1(y)\}_2.\end{aligned}$$

Comparing the coefficients of 1, θ_1, θ_2 and $\theta_1\theta_2$ in the above two expressions we get, for instance, the Poisson brackets (30) to (33). The others can be similarly obtained.

The full set of Poisson brackets is:

$$\{u_1(x), u_1(y)\}_2 = -2\partial_x\delta(x-y) \tag{30}$$
$$\{u_1(x), \psi_1(y)\}_2 = \{\psi_1(x) - 2\psi_2(x)\}\delta(x-y) \tag{31}$$
$$\{\psi_1(x), u_1(y)\}_2 = -\{\psi_1(x) - 2\psi_2(x)\}\delta(x-y) \tag{32}$$
$$\{\psi_1(x), \psi_1(y)\}_2 = \{u_{1,x}(x) - 2u_2(x) - 2\partial_x^2\}\delta(x-y) \tag{33}$$
$$\{u_1(x), \psi_2(y)\}_2 = -\psi_2(x)\delta(x-y) \tag{34}$$
$$\{u_1(x), u_2(y)\}_2 = \{\partial_x^2 - u_1(x)\partial_x - u_{1,x}\}\delta(x-y) \tag{35}$$
$$\{\psi_1(x), \psi_2(y)\}_2 = \{-\partial_x^2 + u_1(x)\partial_x - u_2(x)\}\delta(x-y) \tag{36}$$
$$\{\psi_1(x), u_2(y)\}_2 = -\{\psi_1(x)\partial_x - \psi_2(x)\partial_x - \psi_{1,x}(x)\}\delta(x-y) \tag{37}$$

$$\{\psi_2(x), u_1(y)\}_2 = \psi_2(x)(x-y) \tag{38}$$
$$\{\psi_2(x), \psi_1(y)\}_2 = \{-\partial_x^2 - u_1(x)\partial_x - u_2(x)\}\delta(x-y) \tag{39}$$
$$\{u_2(x), u_1(y)\}_2 = (-\partial_x^2 - u_1(x))\delta(x-y) \tag{40}$$
$$\{u_2(x), \psi_1(y)\}_2 = -\{\psi_1(x)\partial_x + \psi_{2,x}(x) + \psi_2(x)\partial_x\}\delta(x-y) \tag{41}$$
$$\{\psi_2(x), \psi_2(y)\}_2 = 0 \tag{42}$$
$$\{\psi_2(x), u_2(y)\}_2 = -\{2\psi_2(x)\partial_x + \psi_{2,x}(x)\}\delta(x-y)$$
$$\{u_2(x), \psi_2(y)\}_2 = -\{2\psi_2(x)\partial_x + \psi_{2,x}(x)\}\delta(x-y) \tag{43}$$
$$\{u_2(x), u_2(y)\}_2 = -\{2u_2(x)\partial_x + u_{2,x}(x)\}\delta(x-y). \tag{44}$$

Let us define

$$T = -u_2 + \tfrac{1}{2}u_{1,x} \qquad U = u_1 \qquad G^+ = \psi_2 \qquad G^- = \psi_2 - \psi_1. \tag{45}$$

Then we get the following Poisson brackets between these fields:

$$\{T(x), T(y)\}_2 = -\tfrac{1}{2}\partial_x^3\delta(x-y) + 2T(x)\partial_x\delta(x-y) + T_x(x)\delta(x-y) \tag{46}$$
$$\{T(x), U(y)\}_2 = U(x)\partial_x\delta(x-y) \tag{47}$$
$$\{T(x), G^+(y)\}_2 = \tfrac{3}{2}G^+(x)\partial_x\delta(x-y) + \tfrac{1}{2}G_x^+(x)\delta(x-y) \tag{48}$$
$$\{T(x), G^-(y)\}_2 = \tfrac{3}{2}G^-(x)\partial_x\delta(x-y) + \tfrac{1}{2}G_x^-(x)\delta(x-y) \tag{49}$$
$$\{U(x), G^+(y)\}_2 = -G^+(x)\delta(x-y) \tag{50}$$
$$\{U(x), G^-(y)\}_2 = G^-(x)\delta(x-y) \tag{51}$$
$$\{G^+(x), G^+(y)\}_2 = 0 \tag{52}$$
$$\{G^+(x), G^-(y)\}_2 = \{\partial_x^2 - T(x)\}\delta(x-y) + \{U(x)\partial_x + \tfrac{1}{2}U_x(x)\}\delta(x-y) \tag{53}$$
$$\{G^-(x)_1 G^-(y)\}_2 = U_x(x)\delta(x-y). \tag{54}$$

From equations (46)–(49), we can identify T as the energy–momentum tensor, U as a spin 1-field, G^+ and G^- as spin-$\frac{3}{2}$ fields. The above algebra is analogous to the classical $N = 2$ super conformal algebra.

Finally, let us consider a gauge transformation on the operator L:

$$L' = \mathrm{e}^{-\phi}L\mathrm{e}^{\phi}.$$

This L' is then of the form

$$L' = \mathrm{D}^2 + u_{-1} + u_0\mathrm{D}^{-1} + u_1\mathrm{D}^{-2} + \cdots$$

with $u_{-1} = u + \phi^{(2)}$, $u_0 = h$, $u_1 = h\phi^{(1)}$, $u_2 = -h\phi^{(2)}$, and so on.

L' has the form of the even-parity superlax operator of the super KP hierarchy [6]. It should be pointed out that in [6] the $N = 2$, superconformal algebra was also deduced from the super Gelfand Dickey formulae.

6. Discussion

In our above analysis we have demonstrated that an extension of the Lax operator in the sense of an even-order super KP hierarchy can lead to a sensible super extension of dispersive water wave equation. It may be added that an extended form of the dispersive water wave equation was previously given by Kupershmidt himself [7] but it did not involve any odd variables.

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